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RANDOM COVERAGE OF A CIRCLE
WITH APPLICATION TO A SHADOWING PROBLEM

By

M. Yadin and S. Zacks

TECHNICAL REPORT NO. 1

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The coverage problem on the circle is considered from the shadowing process point of view. A random number of shadow arcs are distributed on a circle. The length of each arc is a random variable which depends on the random diameter of a shadowing disk and its random location. Formulae are derived for the numerical determination of the moments of the measure of vacancy of arcs on the circle.

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RANDOM COVERAGE OF A CIRCLE WITH APPLICATION

TO A SHADOWING PROBLEM

1. INTRODUCTION

The problem under consideration in the present study belongs to the class of problems of random coverage of a circle by randomly placed arcs having random length. The specific problem studied here is motivated by a shadowing problem, according to which the random arcs on the circle are shadows cast by randomly scattered disks on the plane. The model assumes that the centers of the disks form a homogeneous Poisson process on the plan, with intensity μ per unit area. Furthermore, given the number of disks centered in a specific Borel set in the plane, the model assumes that their diameters are identically distributed independent (i.i.d) random variables with a known distribution. Assuming that the circle is not intersected by any disk and its center (the source of light) is uncovered, shadow arcs on the circle are defined as the central projections of random disks which lie entirely within the set inscribed by the circle. Accordingly, the model assumes that the number of shadow arcs covering the circle is a Poisson random variable. Given this number, the centers of the disks are conditionally independent having a uniform distribution. The length of the shadow arcs are conditionally i.i.d. random variables, having a common distribution on $[o, \pi]$. The main objective of the present study is to obtain the distribution of a measure of vacancy of arcs on the circle. The measure of vacancy of an arc is the total length of the portion of the arc which is in the light. Equations for the moments and the moment generating function of the measure of vacancy of arcs of length t, $0 < t < 2\pi$, and given in section 4. Furthermore, for $0 < t < \pi$ a formula of the Laplace transform of this moment generating function is developed. The derivations in Section 4 are based on formulae of vacancy probabilities derived in Sections 2 and 3. More specifically, in Section 2 we provide general formulae for: (1) the probability that an arc of length t on the circle is not covered; (2) the probability that a finite set of specified points are simultaneously uncovered. These formulae are further developed in Section 3 in terms of the stochastic specifications of the random arcs. In Section 5 the results are applied to the particular shadowing problem under consideration. Analytic formulae for the numerical determination of the moments of any order, for a particular example, are given in Section 6.

The literature on the coverage problem is very extensive. Robbins [4] derived the moments of the total coverage of an interval on the line by random segments of fixed size. These results were later extended and generalized by Robbins [5], Domb [2], and Takacs [9]. Other related results are presented by Solomon [8]. Siegel presented in [6] moments of the measure of vacancy of the circle when the coverage is by a fixed number of random arcs having random length. In a following paper [7] Siegel provided formulae for the moments and the distribution of the measure vacancy on a circle, which is covered by a given number of random arcs of fixed length. The motivating shadowing problem led us to further developments over those of the previous papers, although the basic approach to the evaluation of moments is essentially the same as that of Robbins.

The shadowing problem did not receive much attention in the literature. Chernoff and Daly [1] considered a similar shadowing problem when the shadows of random disks are cast on a straight line. They provide the methodology for developing the distributions of the length of intervals which are entirely in the light or entirely in shadow. It should be remarked that the shadowing process discussed in the present paper is identical over the interval $[o, \pi]$ with an $M/G/\infty$ queuing process.

2. THE COVERAGE MODEL AND FUNDAMENTAL RESULTS

Consider a circle, C, of radius one centered at the origin. Let $A_1, A_2, \ldots, A_N$ be N arcs placed at random on C. N is a random variable having a Poisson distribution with mean $2\pi\lambda$. Given that $N = n$, the centers of $A_1, \ldots A_n$ are conditionally independent and uniformly distributed on C. Let X_i $(i = 1, \ldots n)$ denote the arc length of A_i. It is assumed that $X_i \ldots, X_n$ are i.i.d. random variables having a c.d.f. $F(x)$ on $[o, \pi]$.

It is well known that the probability of covering any specific point on C by a randomly placed arc is $\gamma = E\{X\}/2\pi$. This result will be obtained as a special case of a more general result derived in the present paper. Let P_τ designate a point on C having polar coordinates $(1, \tau)$, $0 < \tau < 2\pi$. Let $Q(\tau)$ denote the number of arcs which cover P_τ. Given $N = n$, the conditional distribution of $Q(\tau)$ is the binomial $B(n,\gamma)$. Accordingly the (total) distribution of $Q(\tau)$ is the Poisson with mean $\rho = 2\pi\lambda\gamma = \lambda E\{X\}$. Notice that due to the symmetry in the model the distribution of $Q(\tau)$ does not depend on τ.

Consider a specified arc on C of length t, $o<t<2\pi$, connecting the points P_s and P_{s+t}. Let $q_n(t)$ denote the *conditional probability that such an arc is completely* uncovered by random arcs, given $N = n$. This conditional probability is

$$q_n(t) = P\{\bigvee_{s<\tau<s+t} Q(\tau) = 0 | N=n\}, \tag{2.1}$$

where $\bigvee_{\tau} Q(\tau)$ denotes the maximum of $Q(\tau)$ over the specified interval. Since the random arcs are conditionally independent, given $N=n$, $q_n(t) = (q_1(t))^n$, for all t, $o<t<2\pi$. Accordingly, the probability that a specified arc on C of length is uncovered is

$$\begin{aligned} q(t) &= \sum_{n=o}^{\infty} e^{-\lambda 2\pi} \frac{(2\pi\lambda)^n}{n!} (q_1(t))^n \\ &= \exp\{-2\pi\lambda(1-q_1(t))\}. \end{aligned} \tag{2.2}$$

Explicit derivation of $q_1(t)$ is given in Section 3. In particular, for $t=0$, we obtain that $q_1(0)=1-\gamma$ and the probability that any given point on C is uncovered is

$$\xi = q(0) = e^{-\rho} \tag{2.3}$$

Let $P_{s_1},\ldots, P_{s_r}$ be r specified points on C, where $r>2$ and $0<s_1<s_2<\ldots<s_r<2\pi$. Let $t_0= 2\pi-s_r+s_1$, $t_i=s_{i+1}-s_i$, $i=1,\ldots,r-1$, be the length of the arcs connecting these points. The conditional probability, given $N=n$, that all these r points are uncovered is function of $(t_1,\ldots, t_{r-1})$ defined by

$$p_n(t_1,\ldots, t_{r-1}) = P\{\bigvee_{i=1,\ldots r} Q(s_i) = 0 | N=n\} \tag{2.4}$$

$$0<t_i \ (i=1,\ldots, r-1) \text{ and } \sum_{i=1}^{r-1} t_i<2\pi.$$

Again, due to the conditional independence, $p_n(t_1,\ldots, t_{r-1}) = (p_1(t_1,\ldots, t_{r-1}))^n$ for all $(t_1,\ldots, t_{r-1})$. Notice that $t_0 = 2\pi - \sum_{i=1}^{r-1} t_i$. Finally, the (total) probability that the r points are uncovered is

$$p(t_1,\ldots,t_{r-1}) = \exp\{-2\pi\lambda(1-p_1(t_1,\ldots,t_{r-1}))\}. \tag{2.5}$$

Explicit formula for $p_1(t_1,\ldots, t_{r-1})$ will be given in Section 3.

3. VACANCY PROBABILITIES

In the present section we develop explicit formulae for $q_1(t)$ and $p_1(t_1,\ldots,t_{r-1})$. As defined in the previous section, $q_1(t)$ is the probability that a randomly placed arc does not intersect a specified arc of length t. This probability is given by

$$q_1(t) = \frac{1}{2\pi}\int_0^{2\pi-t} F(x)dx. \tag{3.1}$$

Introduce the auxiliary function

$$\phi(t) = \int_t^{2\pi} [1 - F(x)]dx, \ 0<t<2\pi. \tag{3.2}$$

Notice that $\phi(0) = E\{X\}$ and that, $\phi(t)=0$ for all $t>\pi$. Accordingly,

$$q_1(t) = 1 - \frac{1}{2\pi}\left[t + E\{X\} - \phi(2\pi-t)\right], \ 0<t<2\pi. \tag{3.3}$$

From (3.3) one obtains the previously mentioned result that $q_1(0) = 1-E\{X\}/2\pi = 1-\gamma$ and

$$q(t) = \xi\exp\left\{-\lambda\left(t-\phi(2\pi-t)\right)\right\}, \ 0<t<2\pi. \tag{3.4}$$

Another important result that can be obtained from (3.4) is the distribution of the length of an uncovered arc starting at an uncovered point P_s. More specifically, consider the r.v.

$$H_s = \sup \{t; \bigvee_{s<\tau<s+t} Q(\tau) = 0, 0<t<2\pi\} \tag{3.5}$$

where $Q(s) = 0$. It follows that

$$P\{H_s > t\} = q(t)/\xi \tag{3.6}$$

$$= \begin{cases} e^{-\lambda t} & ,0<t<\pi \\ e^{-\lambda t+\lambda\phi(2\pi-t)} & ,\pi<t<2\pi \\ 0 & ,2\pi<t \end{cases}$$

Notice that the distribution of H_s has a jump point at $t=2\pi$ and

$$P\{H_s = 2\pi\} = e^{-2\pi\lambda}/\xi. \tag{3.7}$$

For the derivation of an explicit formula for $p_1(t_1,\ldots, t_{r-1})$, let B_0 denote the event that a single random arc lies entirely within the arc between P_{s_r} and P_{s_1} and B_i ($i=1,\ldots, r-1$) the event that the random arc lies entirely between P_{s_i} and $P_{s_{i+1}}$. Thus,

$$P\{B_i\} = q(2\pi - t_i) \tag{3.8}$$

$$= \frac{1}{2\pi}[t_i + \phi(t_i) - E\{X\}], \; i=0,\ldots, r-1.$$

Finally, since $B_0, \ldots, B_{r-1}$ are mutually exclusive and their union is the event $\{\bigvee_{i=1,\ldots,r} Q(s_i) = 0\}$,

$$p_1(t_1, \ldots, t_{r-1}) = \sum_{i=o}^{r-1} P\{B_i\} \tag{3.9}$$

$$= 1 - \frac{r}{2\pi} E\{X\} + \frac{1}{2\pi} \sum_{i=o}^{r-1} \phi(t_i).$$

Substitution of (3.9) in (2.5) yields

$$p(t_1, \ldots, t_{r-1}) = \xi^r \exp \{\lambda \sum_{i=o}^{r-1} \phi(t_i)\}. \tag{3.10}$$

In particular, for r=2, the joint probability that two specified points, P_s and P_{s+t}, are uncovered by random arcs is

$$p(t) = \begin{cases} \xi^2 e^{\lambda\phi(t)} & ,o<t<\pi \\ \xi^2 & ,t=\pi \\ \xi^2 e^{\lambda\phi(2\pi-t)} & ,\pi<t<2\pi \ . \end{cases} \tag{3.11}$$

4. THE MEASURE OF VACANCY

Define the stochastic process $\{I(t), 0<t\}$ where

$$I(t) = \begin{cases} 0 & ,\text{if } Q(t \bmod 2\pi)>1 \\ 1 & ,\text{if } Q(t \bmod 2\pi)=0. \end{cases} \tag{4.1}$$

Notice that the sample functions of this process are step functions, since with probability one there are only finitely many random arcs. The <u>measure of vacancy</u> of a specified arc from P_s to P_{s+t}, $t>0$, is a r.v. defined as

$$Y(s,t) = \int_s^{s+t} I(\tau)d\tau \quad ,o<s,t\leq 2\pi. \tag{4.2}$$

This measure of vacancy is the sum of lengths of all the uncovered arcs between P_s and P_{s+t}. The distribution of $Y(s,t)$ clearly does not depend on s and is concentrated on the interval $[0,t]$. The corresponding c.d.f. is continuous on $(0,t)$ and has jump points at 0 and t. Furthermore,

$$P\{Y(s,t) = t\} = q(t). \tag{4.3}$$

Let $\xi_r(t)$ denote the r-th moment of $Y(s,t)$. In particular,

$$\xi_1(t) = E\{Y(s,t)\} = \int_0^t E\{I(\tau)\} d\tau = \xi t. \tag{4.4}$$

Indeed, $E\{I(\tau)\} = P\{I(\tau) = 1\} = \xi$.

Furthermore, for every $r\geq 2$ and $0<s_1<s_2<\ldots<s_r<2\pi$,

$$E\{\prod_{i=1}^{r} I(s_i)\} = p(t_1,\ldots, t_{r-1}). \tag{4.5}$$

Accordingly, the r-th moment of $Y(s,t)$, for $r>2$, is

$$\xi_r(t) = r! \underset{s<s_1<\dots<s_r<s+t}{\int \cdots \int} E\left\{\prod_{i=1}^{r} I(s_i)\right\} ds_1 \dots ds_r \tag{4.6}$$

$$= r!\xi^r \int_0^t d\tau\, (t-\tau)e^{\lambda\phi(2\pi-\tau)} \int_0^\tau dt_1 e^{\lambda\phi(t_1)} \int_0^{\tau-t_1} dt_2$$

$$e^{\lambda\phi(t_2)} \int_0^{\tau-t_1-t_2} \cdots \int_0^{\tau-\sum_{i=1}^{r-3} t_i} dt_{r-2} \exp\{\lambda\phi(t_{r-2}) + \lambda\phi(\tau - \sum_{i=1}^{r-2} t_i)\}.$$

Let $\{\Psi_r(t);r>1\}$ be a sequence of convolutions defined recursively in the following manner:

$$\psi_1(t) = \begin{cases} 0 & ,\text{if } t<0 \\ e^{\lambda\phi(t)} & ,\text{if } t>0 \end{cases} \tag{4.7}$$

and

$$\psi_r(t) = \int_0^t \psi_1(\tau)\psi_{r-1}(t-\tau)d\tau,\ r>2.$$

Accordingly, formula (4.6) can be expressed in the form

$$\xi_r(t) = r!\xi^r \int_0^t (t-\tau)\psi_1(2\pi-\tau)\psi_{r-1}(\tau)d\tau. \tag{4.8}$$

From formula (4.8) we obtain immediately that

$$\frac{d}{dt}\ \xi_r(t) = r!\xi^r \int_0^t \psi_1(2\pi-\tau)\psi_{r-1}(\tau)d\tau \tag{4.9}$$

and

$$\frac{d^2}{dt^2}\ \xi_r(t) = r!\xi^r\psi_1(2\pi-t)\psi_{r-1}(t). \tag{4.10}$$

This means that, for every $r>2$, $\xi(t)$ is an increasing convex function of t and for $r=1$ it is an increasing linear function of t.

Introduce the moment generating function (m.g.f.) of $Y(s,t)$

$$\xi(\nu,t) = \sum_{r=0}^{\infty} \frac{\nu^r}{r!}\ \xi_r(t),\ -\infty<\nu<\infty\ , \tag{4.11}$$

and let

$$\psi(\nu,t) = \sum_{r=1}^{\infty} \nu^r\ \psi_r(t)\quad ,\ -\infty<\nu<\infty\ . \tag{4.12}$$

Notice that $\phi(t) < E\{X\}$ and therefore $1 < \psi_1(t) < 1/\xi$. It follows that $\psi_r(t) = O(\frac{t^r}{r!})$, $r \geq 1$ and therefore $\psi(\nu,t)$ is convergent for all real ν. Furthermore, from (4.7), the generating function $\psi(\nu,t)$ satisfies the renewal equation.

$$\psi(\nu,t) = \nu\psi_1(t) + \nu \int_0^t \psi_1(\tau)\, \psi(\nu,t-\tau) d\tau. \tag{4.13}$$

Finally, from (4.8), (4.11), and (4.12) we obtain the formula

$$\xi(\nu,t) = 1 + \xi t\nu + \xi\nu \int_0^t (t-\tau)\, \psi_1(2\pi-\tau)\, \psi(\xi\nu,\tau)\, d\tau\ . \tag{4.14}$$

Generally, the distribution function of $Y(s,t)$ can be obtained from (4.14). We provide now further development for the case of $t<\pi$. In this case $\psi(2\pi-\tau) = 1$ for all $0<\tau<t$. Hence, from formulae (4.7) and (4.8) one obtains the recursive formula

$$\xi_r(t) = r\xi \int_0^t \psi_1(u)\xi_{r-1}(t-u)du\ ,\ r \geq 2\ . \tag{4.15}$$

It follows that the m.g.f of $Y(s,t)$ satisfies the integral equation

$$\xi(\nu,t) = \gamma(\nu,t) + \xi\nu \int_0^t \psi_1(u)\xi(\nu,t-u)du, \tag{4.16}$$

where

$$\gamma(\nu,t) = 1 + \xi\nu(t - \int_0^t \psi_1(u)du)\ . \tag{4.17}$$

Let $h(\nu,t)$ be the solution of (4.16) for all real ν and non-negative t. Clearly, for $0<t<\pi$, $h(\nu,t) = \xi(\nu,t)$. This solution can be interpreted as the m.g.f. of the measure of vacancy of a specified interval of length t on the real line, when the coverage process is by random intervals of length X, having a c.d.f. F(x). Let $h^*(\nu,\omega)$, $\omega>0$, be the Laplace transform of $h(\nu,t)$. From (4.16) one obtains

$$h^*(\nu,\omega) = \frac{\gamma^*(\nu,\omega)}{1 - \xi\nu\Psi^*(\omega)} , \tag{4.18}$$

where $\gamma^*(\nu,\omega)$ and $\Psi^*(\omega)$ are the Laplace transforms of $\gamma(\nu,t)$ and $\psi_1(t)$, respectively. Furthermore,

$$\gamma^*(\nu,\omega) = \frac{\xi\nu}{\omega^2} + \frac{1}{\omega}\left(1 - \xi\nu\Psi^*(\omega)\right) \tag{4.19}$$

Hence,

$$h^*(\nu,\omega) = \frac{1}{\omega} + \frac{\xi\nu}{\omega^2} \cdot \frac{1}{1-\xi\nu\Psi^*(\omega)} \tag{4.20}$$

5. APPLICATION TO A SHADOWING PROBLEM

Consider a countable number of randomly distributed disks on the plane. For any Borel set B in the plane, the number of disks, N{B}, centered in B is a random variable having a Poisson distribution with mean $\mu m\{B\}$ where μ is the average number of disks per unit area and m{B} is the 2-dimensional Lebesgue measure of B (area). We further assume that, given N{B}=n, the centers of these n disks are conditionally independent and uniformly distributed over B. The diameters of the disks are i.i.d. random variables, Y_1, Y_2,... having a common c.d.f. G(y), $0<y<\infty$. Let C be a circle in the plane which does not intersect any one of the random disks and whose center, O, is uncovered. The central projections on C of disks whose centers lie within C will be called shadow-arcs. The results of the previous sections are applied to determine properties of the distributions of the measure of vacancy of specified arcs on C.

Let (ρ,θ) denote the polar coordinates of the center of a disk, with respect to 0. Thus, each random disk is specified by a triplet of random variables (ρ,θ,y). We consider only disks with random parameter vector (ρ,θ,y) in the set

$$S = \{(\rho,\theta,y);\ \frac{y}{2} < \rho < 1-y/2;\ 0 \leq \theta < 2\pi,\ 0 < y < 1\} \tag{5.1}$$

$N\{S\}$ has a Poisson distribution with mean

$$E\{N\{S\}\} = \mu \int_0^1 \int_0^{2\pi} \int_{y/2}^{1-y/2} \rho d\rho d\theta dG(y) = 2\pi\lambda, \tag{5.2}$$

where

$$\lambda = \frac{\mu}{2} \int_0^1 (1-y)dG(y) . \tag{5.3}$$

It is assumed that $G(y)$ is absolutely continuous with p.d.f $g(y)$. The conditional p.d.f. of (ρ,θ,y) within S is

$$f_S(\rho,\theta,y) = \frac{\mu\rho g(y)}{2\pi\lambda} \qquad ,(\rho,\theta,y)\varepsilon S \tag{5.4}$$

Let $X(\rho,\theta,y)$ denote the length of a shadow-arc projected by a disk with parameters (ρ,θ,y). This length is given by

$$X(\rho,\theta,y) = 2 \sin^{-1} (y/2\rho) \tag{5.5}$$

Given $N\{S\} = n$, let $X_1,\ldots,X_n$ designate i.i.d. random variables representing the lengths of the n shadow-arcs. The common distribution of these random variables is the c.d.f., $F(x)$, of the random arcs discussed in the previous sections. According to (5.4) and (5.5)

$$F(x) = \frac{\mu}{\lambda} \int_0^{D(x)} \left(\int_{\frac{y}{2\sin(x/2)}}^{1-y/2} \rho d\rho \right) g(y)dy \qquad (5.6)$$

$$= \frac{\mu}{2\lambda} \int_0^{D(x)} \left(1-y-\frac{y^2}{4}\cot^2(x/2)\right) g(y)dy,$$

where

$$D(x) = 2\sin(x/2)/(1+\sin(x/2)) = 1-\tan^2\left(\frac{\pi-x}{4}\right) \qquad (5.7)$$

is the largest diameter of a disk that can yield a shadow-arc of length x. That is, if $y < D(x)$ then $y/(2\sin(x/2)) < 1-y/2$. Consider the distribution functions, related to G(x),

$$B_\ell(x) = \frac{1}{c_\ell} \int_0^{D(x)} y^\ell (1-y)^{1-\ell/2} g(y)dy, \quad 0<x<\pi \qquad (5.8)$$

where

$$c_\ell = \int_0^1 y^\ell (1-y)^{1-\ell/2} g(y)dy$$

is the normalization constant, Notice that $c_0=2\lambda/\mu$. The distribution function F(x) can then be expressed as

$$F(x) = \begin{cases} 0 & ,x<0 \\ B_0(x) - \frac{1}{4}\frac{c_2^{\,2}}{c_0} B_2(x)\cot^2(x/2) & ,0<x<\pi \; . \\ 1 & ,\pi<x \end{cases} \qquad (5.9)$$

It should be remarked that

$$\frac{1}{2} \cot (x/2) = \sqrt{1-D(x)}/D(x) \cdot \tag{5.10}$$

Hence, since $D(x) \to 0$ (5.8) yields

$$\frac{1}{4} \cot^2 (x/2)\, B_2(x) = c_2(1-D(x))G(D(x)) + o(D(x)). \tag{5.11}$$

and $\cot^2(x/2)B_2(x) \to 0$ as $x \to 0$. Furthermore, since $\cot^2(x/2) = \frac{d}{dx}\,(x+2\cot(x/2))$ one obtains from (3.2), (5.9), and integration by parts

$$\phi(t) = \int_t^\pi [1-B_0(x)]dx + \frac{c_2}{4c_0} \int_t^\pi [1-B_2(x)]dx \tag{5.12}$$

$$+ \frac{1}{c_0}\left(c_1 - \frac{c_2}{4}\pi\right) + \frac{c_2}{2c_0}\left[\cot (t/2)\, B_2(t) + (t/2)\right]$$

$$- \frac{c_1}{c_0} B_1(t).$$

6. COMPUTING THE MOMENTS IN A SPECIAL EXAMPLE

In the present section we apply the theory of the previous section to develop formulae for the determination of the moments of the measure of vacancy in the special case that $G(y)$ is the uniform distribution on $(0,1)$. In this case we obtain according to (5.7) and (5.8) the formulae

$$\int_t^\pi [1-B_0(x)]dx \tag{6.1}$$

$$= \int_t^\pi \tan^4 \left(\frac{\pi-4}{4}\right) dx = \frac{4}{3} \tan^3\left(\frac{\pi-t}{4}\right) -4\tan \left(\frac{\pi-t}{4}\right) + (\pi-t) ,$$

$$B_1(t) = 1- \frac{5}{2} \tan^3\left(\frac{\pi-t}{4}\right) + \frac{3}{2} \tan^5\left(\frac{\pi-t}{4}\right) \tag{6.2}$$

and

$$\int_t^\pi [1-B_2(x)]dx = \frac{4}{5} \tan^5\left(\frac{\pi-t}{4}\right) - \frac{16}{3} \tan^3\left(\frac{\pi-t}{4}\right) \tag{6.3}$$

$$+ 28\tan\left(\frac{\pi-t}{4}\right) -7(\pi-t) .$$

Furthermore,

$$\frac{1}{2} \cot (t/2) B_2(t) = D^2(t)(1-D(t))^{1/2} \tag{6.4}$$

$$= \tan \left(\frac{\pi-t}{4}\right) -2\tan^3\left(\frac{\pi-t}{4}\right) + \tan^5\left(\frac{\pi-t}{4}\right) .$$

Finally, since $c_0=1/2$, $c_1=4/15$ and $c_2=1/3$ one obtains by substituting the above results in (5.12) the special formula for $\phi(t)$

$$\phi(t) = - \frac{1}{3} (\pi-t) + \frac{4}{3} \tan \left(\frac{\pi-t}{4}\right) + \frac{4}{9} \tan^3\left(\frac{\pi-t}{4}\right) . \tag{6.5}$$

Accordingly, in the case of uniformly distributed diameters on (0,1), the expected length of a shadow arc is $E\{X\} = \phi(0) = -\frac{\pi}{3} + \frac{4}{3} + \frac{4}{9} = 0.7306$, and the first moment of vacancy of an arc of length t is $\xi_1(t) = t \cdot \exp\{-0.7306\lambda\}$.

In order to determine higher moments $\xi_r(t)$, $r>2$, we have to perform the convolutions (4.7) and (4.8) recursively, where

$$\psi_1(t) = \begin{cases} 0 & ,t<0 \\ \exp\{-\frac{4}{3}\lambda[(\frac{\pi-t}{4}) - \tan(\frac{\pi-t}{4}) - \frac{1}{3}\tan^3(\frac{\pi-t}{4})]\}, & 0<t<\pi \\ 1 & ,\pi<t \end{cases} \quad (6.6)$$

We present now a polynomial approximation to the moments $\xi_r(t)$, $r>2$, for $0<t<\pi$. In this range of t values the moments will be approximated by an analytic solution of the recursive equation,

$$\tilde{\xi}(t) = \xi_1(t)$$

$$\tilde{\xi}(t) = r\xi \int_0^t \tilde{\psi}(u)\tilde{\xi}_{r-1}(t-u),\ r>2, \quad (6.7)$$

where $\tilde{\psi}(u)$ is a polynomial approximating $\psi_1(u)$, over $[0,\pi]$. Notice that $\psi_1(u)$ is an analytic function and can be approximated by a polynomial of a proper degree. We approximate the function (6.6) by the fourth degree polynomial

$$\tilde{\psi}(t) = 2.0486 - 1.69t + 1.124t^2 - .3489t^3 + 0.0411t^4 \quad (6.8)$$

The coefficients b_i of t^i $(i=0,\ldots,4)$ in (6.8) were determined by the method of least squares by fitting a fourth degree polynomial to 33 points $(t_i, \psi_1(t_i))$, where $t_i = i\pi/32$, $i=0,\ldots,32$. The standard deviation of the residuals $\tilde{\psi}_1(t_i) - \psi(t_i)$, with 28 degrees of freedom is $\sigma=.00911$, with a squared-multiple correlation of $R^2=.999$. This is a very high degree of accuracy in approximating $\psi_1(t)$ by $\tilde{\psi}(t)$. Define recursively the coefficients,

$$C_{1,i} = \begin{cases} 1 & , i=0 \\ 0 & , i>0 \end{cases} , \qquad (6.9)$$

and for each r>2

$$C_{r,i} = \begin{cases} \sum_{j=0}^{i} b_j C_{r-1,i-j} / \binom{i}{j} & , i=0,\ldots,4(r-1) \\ \\ 0 & , i>4(r-1) \end{cases} \qquad (6.10)$$

in which

$$b_j = \begin{cases} \text{coefficients of (6.8)} & ,j=0,\ldots,4 \\ 0 & ,j>4. \end{cases} \qquad (6.11)$$

Qne can prove then, by induction on r, that

$$\xi_r(t) \cong t^r \xi^r \sum_{i=0}^{4(r-1)} \frac{C_{r,i}}{\binom{r+i}{r}} t^i \qquad , r > 1 \qquad (6.12)$$

For small arcs, i.e., as $t \to 0$, formula (6.12) can be simplified, by approximating $\psi_1(t)$ by $\hat{\psi}(t)=\psi_1(0)+t\psi_1'(0)$, where the derivitave of $\psi_1(t)$ is

$$\psi_1'(t) = \frac{\lambda}{3} \psi_1(t) \left(1 - \frac{1}{\cos^4 \left(\frac{\pi - t}{4}\right)}\right) , 0 < t < \pi. \qquad (6.13)$$

Thus, as $t \to 0$ we obtain the approximation

$$\hat{\xi}_r(t) = tr\xi r \sum_{j=0}^{r-1} \frac{\binom{r-1}{j}}{j!\binom{r+1}{j}} \psi_1^{r-1-j} (\psi_1')^j t^j \ , \tag{6.14}$$

where $\psi_1 = \psi_1(0)$ and $\psi_1' = \psi_1'(0)$. In Table 1 we provide numerical values of the normalized moments $\tilde{\xi}_r(t)/t^r$, for $r=2,\ldots,10$ and the limiting value

$$\lim_{r\to\infty} \xi_r(t)/t^r = q(t) \ , \tag{6.15}$$

for values of $t_i = i\pi/M$, $i=4,8,\ldots,64$ $(M=64)$. The values of the normalized moments for the case of $i=4$ were computed according to (6.14), with $\psi_1(0)=2.076326$ and $\psi_1'(0) = -\psi_1(0)$. The normalized moment of order 1 is $\xi = .48162$ for all t. In Table 2 we present the corresponding standard-deviations $\sigma(t_i)$, measures of skewnesss, $\gamma_1(t_i)$ and kurtosis $\gamma_2(t_i)$, where

$$\begin{aligned} \gamma_1(t_i) &= \mu_3(t_i)/\sigma^3(t_i) \\ \gamma_2(t_i) &= \mu_4(t_i)/\sigma^4(t_i) \ , \end{aligned} \tag{6.16}$$

and $\mu_3(t_i)$, $\mu_4(t_i)$ are the third and fourth central moments. According to Table 2, the distributions of the measure of vacancy are for small arcs (as $t\to 0$) negatively skewed and sharply increasing near the right limit of the interval. On the other hand, as t increases to π the distributions become more symmetric and can be approximated within $(0,t)$ by Pearson's Type I distributions (see Johnson and Kotz [3]).

Table 1. The Normalized Moments $\tilde{\xi}_r(t)/t^r$ Determined According to (6.12) and (6.14), for $t_i=i\pi/64$ ($i=4,8,\ldots,64$)

i/r	2	3	4	5	6	7	8	9	10	∞
4	0.4816	0.4343	0.4267	0.4212	0.4173	0.4145	0.4123	0.4105	0.4091	0.3958
8	0.4301	0.4024	0.3842	0.3707	0.3599	0.3508	0.3428	0.3356	0.3291	0.3252
12	0.4117	0.3755	0.3525	0.3362	0.3236	0.3133	0.3046	0.2970	0.2901	0.2672
16	0.3957	0.3520	0.3250	0.3062	0.2921	0.2809	0.2715	0.2636	0.2565	0.2196
20	0.3817	0.3315	0.3010	0.2801	0.2647	0.2527	0.2429	0.2346	0.2275	0.1804
24	0.3695	0.3136	0.2800	0.2574	0.2409	0.2282	0.2180	0.2096	0.2023	0.1483
28	0.3589	0.2980	0.2617	0.2375	0.2201	0.2069	0.1964	0.1878	0.1805	0.1218
32	0.3497	0.2843	0.2457	0.2201	0.2020	0.1883	0.1775	0.1688	0.1615	0.1001
36	0.3415	0.2723	0.2315	0.2049	0.1860	0.1720	0.1611	0.1522	0.1449	0.0823
40	0.3343	0.2616	0.2191	0.1914	0.1720	0.1577	0.1466	0.1378	0.1305	0.0676
44	0.3280	0.2522	0.2081	0.1795	0.1597	0.1451	0.1339	0.1251	0.1178	0.0556
48	0.3223	0.2438	0.1983	0.1690	0.1488	0.1341	0.1228	0.1139	0.1068	0.0456
52	0.3172	0.2363	0.1895	0.1596	0.1391	0.1242	0.1129	0.1041	0.0970	0.0375
56	0.3125	0.2295	0.1817	0.1512	0.1304	0.1154	0.1042	0.0954	0.0884	0.0308
60	0.3083	0.2233	0.1746	0.1437	0.1227	0.1077	0.0964	0.0877	0.0808	0.0253
64	0.3045	0.2178	0.1682	0.1369	0.1158	0.1007	0.0895	0.0809	0.0741	0.0208

Table 2. Standard Deviations and Measures of Skewness and Kurtosis

i	σ_i	$\gamma_{1,i}$	$\gamma_{2,i}$
4	0.09811	-0.30546	1.58630
8	0.17480	0.05021	1.17473
12	0.24975	0.05300	1.26210
16	0.31780	0.05611	1.34884
20	0.37993	0.05902	1.43394
24	0.43699	0.06130	1.51655
28	0.48976	0.06262	1.59601
32	0.53389	0.06290	1.67165
36	0.58490	0.06221	1.74299
40	0.62823	0.06089	1.80958
44	0.66921	0.05942	1.87096
48	0.70808	0.05835	1.92677
52	0.74505	0.05814	1.97704
56	0.78027	0.05897	2.02223
60	0.81398	0.06057	2.06333
64	0.84627	0.06179	2.10328

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